THEORY OF MAGNETOACOUSTIC WAVE GENERATION BY MECHANICAL RADIATORS

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Aspects of the radiation of magnetoacoustic waves by an oscillating plane piston and by a radially pulsating cylinder and sphere are discussed. Expressions are derived for the magnetohydrodynamic perturbation fields in the medium. The radiation reaction forces acting on the given bodies are determined. The significant distinction between the theory of so-called "magnetic" sound generation and the theory of ordinary sound radiation is demonstrated. For example, the directivity pattern of a magnetic radially pulsating cylinder or sphere has a dipole character, whereas for ordinary sound generated by the same sources the pattern is isotropic, i.e., monopolar.

Magnetohydrodynamic processes are described by a system of coupled hydrodynamic and electromagnetic equations [1]. As a result, hydromagnetic perturbations can be created either by mechanical means, i.e., by oscillating or pulsating bodies, or by electrical charges and currents. The radiation of hydromagnetic waves by electrical currents has been analyzed in detail in [2-5]. We now investigate the generation of such waves by mechanical radiators. We find the radiation reaction forces acting on the wave sources, using the method of force sources [6].

1. Integral Form of the Solution of the Linear Magnetohydrodynamic Equations

The system of linear magnetohydrodynamic equations for an inviscid medium in the presence of external forces with a spatial density f has the form

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \frac{1}{4\pi} (\mathrm{rot}\, \mathbf{h}) \times \mathbf{H}_0 + \mathbf{f} \tag{1.1}$$

$$\frac{\partial \mathbf{h}}{\partial t} = \mathrm{rot}\,(\mathbf{v} \times \mathbf{H}_0) \tag{1.2}$$

$$\frac{\partial \rho}{\partial t} + \rho_0 \,\mathrm{div}\, \mathbf{v} = 0 \tag{1.3}$$

$$p = c_S^2 \rho \tag{1.4}$$

Here ρ_0 and $\mathbf{H}_0$ are the unperturbed density and magnetic field in the medium; c_S is the speed of sound; and ρ, p, $\mathbf{v}$, and $\mathbf{h}$ are the density, pressure, velocity, and magnetic field perturbations. We assume that all the perturbations vary harmonically with time as $\exp(-i\omega t)$, where ω is the frequency of the process. In this case we readily obtain the following equation for the velocity from the system (1.1)-(1.4):

$$c_S^2\, \mathrm{grad\, div}\, \mathbf{v} + c_A^2 [\mathrm{rot\, rot}\,(\mathbf{v} \times \mathbf{e}_H)] \times \mathbf{e}_H + \omega^2 \mathbf{v} = i\omega \mathbf{f}/\rho_0 \tag{1.5}$$

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where $c_A = H_0/(4\pi\rho_0)^{1/2}$ is the Alfvén velocity, and e_H is the unit vector of the field H_0. From now on we drop the time factor $\exp(-i\omega t)$ from the functions $\mathbf{v}$ and $\mathbf{f}$ for brevity of notation.

The spatial density of forces $\mathbf{f}$ is expressed by the formalism of generalized functions in terms of the grazing surface force vector:

$$\mathbf{f} = \mathbf{T}\delta_S \tag{1.6}$$

where δ_S is the Dirac delta function, whose argument is the equation for the surface of the body [7]. The integral of $\mathbf{f}$ over the normal to the surface yields the function $\mathbf{T}$. By the laws of mechanics $(-\mathbf{T})$ is the force exerted on the body by the medium (the reaction force of the medium). Equation (1.5) refers to right-sided vector differential operators, whose analysis is considerably more complex than for scalar operators [8].

In an inviscid medium the normal component of the velocity at the surface of the body must be equal to the corresponding velocity component of the body [9]:

$$\mathbf{v}|_S = u_0\mathbf{n} \tag{1.7}$$

where $\mathbf{n}$ is the outward normal unit vector relative to the surface of the body. It is assumed here that the bodies creating the perturbations in the medium are metals (whose conductivities are large in comparison with that of the surrounding medium), so that $(\mathbf{v}-\mathbf{u}_0)_S \mathrm{x} \cdot \mathbf{n} = 0$.

We formulate the fundamental problem in application to relations (1.5)-(1.7). It is required to determine the surface force $\mathbf{T}$ (1.6) from the known differential operator (1.5) subject to the boundary condition (1.7). Thus stated, the problem belongs to the class of inverse problems of the theory of differential equations, wherein the right-hand side of an equation is to be determined from a known operator and boundary conditions [10]. Inverse problems of this type are ordinarily reduced to integral equations for the unknown right-hand side of the primary differential equation. For example, in the theory of subsonic flow past slender bodies an integral equation is obtained for the distribution of mass sources and sinks.

The wave radiation intensity I is related to the force $\mathbf{T}$ by the simple expression

$$I = -{}^1/_2(\mathbf{v}\mathbf{T}^*)_S \tag{1.8}$$

in which $\mathbf{T}^*$ denotes the complex conjugate.

To solve the problem we use the Fourier transformations

$$\mathbf{v}_1(\mathbf{k}) = \frac{1}{8\pi^3}\iiint\limits_{-\infty}^{+\infty} \mathbf{v}(\mathbf{r})\, e^{i(\mathbf{kr})}\, d\mathbf{r}, \quad \mathbf{f}_1(\mathbf{k}) = \frac{1}{8\pi^3}\iiint\limits_{-\infty}^{+\infty} \mathbf{f}(\mathbf{r})\, e^{i(\mathbf{kr})}\, d\mathbf{r} \tag{1.9}$$

and the corresponding inversion formulas

$$\mathbf{v}(\mathbf{r}) = \iiint\limits_{-\infty}^{+\infty} \mathbf{v}_1(\mathbf{k})\, e^{-i(\mathbf{kr})}\, d\mathbf{k}, \quad \mathbf{f}(\mathbf{r}) = \iiint\limits_{-\infty}^{+\infty} \mathbf{f}_1(\mathbf{k})\, e^{-i(\mathbf{kr})}\, d\mathbf{k} \tag{1.10}$$

$$d\mathbf{r} = dx\,dy\,dz, \quad d\mathbf{k} = dk_x\,dk_y\,dk_z, \quad (\mathbf{kr}) = k_x x + k_y y + k_z z$$

in which $\mathbf{v}_1(\mathbf{k})$ and $\mathbf{f}_1(\mathbf{k})$ are the Fourier transforms of the velocity and force density. It is assumed that the functions $\mathbf{v}$ and $\mathbf{f}$ admit Fourier transformation in generalized function spaces [7, 11]. We choose the z axis of our coordinate system (x, y, z) in the direction of the magnetic field $\mathbf{H}$ ($e_z = e_H$). Applying the transformation (1.9) to Eq. (1.5), we obtain a vector algebraic equation for $\mathbf{v}_1(\mathbf{k})$, from which we find the components of that vector. Using the inversion formulas (1.10), we obtain the following integral form of the solution of Eq. (1.5):

$$\rho_0 v_x = i\omega \iiint\limits_{-\infty}^{+\infty}\left[\frac{k_y(k_y f_{1x} - k_x f_{1y})}{(k_x^2 + k_y^2)D_A} + \frac{k_x(\omega^2 - c_S^2 k_z^2)(k_x f_{1x} + k_y f_{1y})}{(k_x^2 + k_y^2)D_S} + \frac{c_S^2 k_x k_z f_{1z}}{D_S}\right] e^{-i(\mathbf{kr})} d\mathbf{k} \tag{1.11}$$

$$\rho_0 v_y = i\omega \iiint_{-\infty}^{+\infty} \left[\frac{k_x(k_y f_{1y} - k_y f_{1x})}{(k_x^2 + k_y^2) D_A} + \frac{k_y(\omega^2 - c_S^2 k_z^2)(k_x f_{1x} + k_y f_{1y})}{(k_x^2 + k_y^2) D_S} + \frac{c_S^2 k_y k_z f_{1z}}{D_S} \right] e^{-i(\mathbf{kr})} \, d\mathbf{k} \tag{1.12}$$

$$\rho_0 v_z = i\omega \iiint_{-\infty}^{+\infty} \frac{[\omega^2 - (c_A^2 + c_S^2)(k_x^2 + k_y^2 + k_z^2)] f_{1z} + c_S^2 k_z (\mathbf{kf}_1)}{D_S} \, e^{-i(\mathbf{kr})} \, d\mathbf{k} \tag{1.13}$$

The following notation is used above:

$$D_A = \omega^2 - c_A^2 k_z^2, \quad D_S = \omega^4 - [\omega^2 (c_A^2 + c_S^2) - c_A^2 c_S^2 k_z^2] (k_x^2 + k_y^2 + k_z^2)$$

The conditions $D_A = 0$ and $D_S = 0$ yield dispersion relations for Alfvén and magnetoacoustic waves with phase velocities

$$u_A = c_A \cos\alpha, \quad u_\pm^2 = \frac{c_A^2 + c_S^2 \pm \sqrt{(c_A^2 + c_S^2)^2 - 4 c_A^2 c_S^2 \cos^2\alpha}}{2} \tag{1.14}$$

where α is the angle between the vectors $\mathbf{k}$ and $\mathbf{H}_0$, and $u_\pm$ are the respective speeds of the fast and slow magnetoacoustic waves. A complete analysis of the dispersion relations and their corresponding wave surfaces may be found in [12].

For the plane problem, in which $\mathbf{f} = \mathbf{f}\,(x, z)$ and $\mathbf{v} = \mathbf{v}(x, z)$ we find the following from (1.11)-(1.13):

$$\rho_0 v_x = i\omega \iint_{-\infty}^{+\infty} \frac{(\omega^2 - c_S^2 k_z^2) f_{2x} + c_S^2 k_x k_z f_{2z}}{\omega^4 - [(c_A^2 + c_S^2)\omega^2 - c_A^2 c_S^2 k_z^2](k_x^2 + k_z^2)} \, e^{-ik_x x - ik_z z} dk_x \, dk_z \tag{1.15}$$

$$\rho_0 v_z = i\omega \iint_{-\infty}^{+\infty} \frac{c_S^2 k_x k_z f_{2x} + [\omega^2 - c_S^2 k_x^2 - c_A^2 (k_x^2 + k_z^2)]}{\omega^4 - [(c_A^2 + c_S^2)\omega^2 - c_A^2 c_S^2 k_z^2](k_x^2 + k_z^2)} \, e^{-ik_x x - ik_z z} \, dk_x \, dk_z \tag{1.16}$$

$$\rho_0 v_y = 0$$

Here we have made use of the method of integral descent on the y coordinate [7, 11]:

$$\mathbf{f}_1(\mathbf{k}) = \frac{1}{8\pi^3} \iiint_{-\infty}^{+\infty} \mathbf{f}\,(x, z)\, e^{i\,(\mathbf{kr})} \, d\mathbf{r} = f_2(\mathbf{k})\, \delta\,(k_y) \tag{1.17}$$

$$\mathbf{f}_2\,(k) = \frac{1}{4\pi^2} \iint_{-\infty}^{+\infty} \mathbf{f}\,(x, z)\, e^{i(k_x x + k_z z)} \, dx \, dz \tag{1.18}$$

The integral forms of the solution (1.11)-(1.16), (1.17), (1.18) are used below to analyze problems in the generation of magnetoacoustic waves.

2. Radiation by an Oscillating Piston

We consider the radiation of magnetoacoustic waves by a plane oscillating harmonically at a frequency ω in the direction of the normal to the surface. The normal forms an angle α with the z axis, along which the force lines of the field $\mathbf{H}_0$ are directed (Fig. 1). The zx plane contains the normal to the surface. The boundary condition (1.7) has the following form in projections onto the x and z axes:

$$v_x|_S = -u_0 \sin\alpha, \quad v_z|_S = u_0 \cos\alpha \tag{2.1}$$

The s plane is given by the equation

$$x \sin\alpha - z \cos\alpha = 0 \tag{2.2}$$

Invoking the theory of generalized functions, we express the distribution of the spatial force density $\mathbf{f}$ in terms of the surface force $\mathbf{T}$:

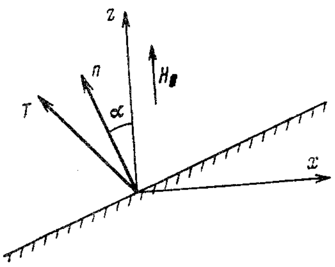

Fig. 1

$$\mathbf{f} = (T_x \mathbf{e}_x + T_z \mathbf{e}_z)\,\delta(x\sin\alpha - z\cos\alpha) \tag{2.3}$$

where T_x and T_z are the projections of the unknown force **T** exerted by unit area of the piston surface on the medium, $\mathbf{e}_x$ and $\mathbf{e}_z$ are the unit vectors in the x and z directions, and δ is the Dirac delta function. Due to the magnetic anisotropy of the properties of the medium, the force T in general does not coincide in direction with the normal to the surface.

The Fourier transform of the distribution (2.3) is found from Eq. (1.18):

$$\mathbf{f}_2(\mathbf{k}) = \frac{T_x \mathbf{e}_x + T_z \mathbf{e}_z}{2\pi}\,\delta(k_z\sin\alpha + k_x\cos\alpha) \tag{2.4}$$

This means that the Fourier transform of the function f with respect to x and z exists only in the sense of generalized functions, having the form of the delta-distrubution function (2.4) in **k** representation. Substituting expression (2.4) into Eqs. (1.15) and (1.16) and integrating on k_z, we obtain

$$v_x = \frac{i\omega\sin\alpha}{2\pi\rho_0}\int_{-\infty}^{+\infty} \frac{(\omega^2 - c_S^2k^2\,\mathrm{ctg}^2\,\alpha)\,T_x - c_S^2k^2T_z\,\mathrm{ctg}\,\alpha}{\omega^4\sin^2\alpha - [(c_A^2 + c_S^2)\,\omega^2 - k^2c_A^2c_S^2\,\mathrm{ctg}^2\,\alpha]\,k^2}\, e^{-ik(x - z\,\mathrm{ctg}\,\alpha)}\,dk \tag{2.5}$$

$$v_y = \frac{i\omega\sin\alpha}{2\pi\rho_0}\int_{-\infty}^{+\infty} \frac{[\omega^2 - (c_S^2 + c_A^2\,\mathrm{cosec}^2\,\alpha)\,k^2]\,T_z - c_S^2k^2T_x\,\mathrm{ctg}\,\alpha}{\omega^4\sin^2\alpha - [(c_A^2 + c_S^2)\,\omega^2 - c_A^2c_S^2k^2\,\mathrm{ctg}^2\,\alpha]\,k^2}\, e^{-ik(x - z\,\mathrm{ctg}\,\alpha)}\,dk \tag{2.6}$$

Here we have dropped the subscript x from the wave number k_x. The improper integrals in (2.5) and (2.6) are most readily computed in the complex plane of the variable k. The rules for bypassing the poles of the integrands are adopted in accordance with the radiation principle, i.e., so that the solution with inclusion of the time factor $\exp(-i\omega t)$ will have the form of plane waves traveling into the left half-space from the plane (2.2):

$$v_x = \frac{(u_+^2 - c_S^2\cos^2\alpha)\,T_x - c_S^2T_z\sin\alpha\cos\alpha}{\rho_0 u_+ (u_+^2 - u_-^2)} \exp\left[-i\omega\left(t + \frac{x\sin\alpha - z\cos\alpha}{u_+}\right)\right] - \frac{(u_-^2 - c_S^2\cos^2\alpha)\,T_x - c_S^2T_z\sin\alpha\cos\alpha}{\rho_0 u_- (u_+^2 - u_-^2)} \exp\left[-i\omega\left(t + \frac{x\sin\alpha - z\cos\alpha}{u_-}\right)\right] \tag{2.7}$$

$$v_z = \frac{(u_+^2 - c_A^2 - c_S^2\sin^2\alpha)\,T_z - c_S^2T_x\sin\alpha\cos\alpha}{\rho_0 u_+ (u_+^2 - u_-^2)} \exp\left[-i\omega\left(t + \frac{x\sin\alpha - z\cos\alpha}{u_+}\right)\right] - \frac{(u_-^2 - c_A^2 - c_S^2\sin^2\alpha)\,T_z - c_S^2T_x\sin\alpha\cos\alpha}{\rho_0 u_- (u_+^2 - u_-^2)} \exp\left[-i\omega\left(t + \frac{x\sin\alpha - z\cos\alpha}{u_-}\right)\right] \tag{2.8}$$

In relations (2.7) and (2.8) $u_\pm$ denotes the phase velocities of the fast and slow waves (1.14).

For the determination of the unknowns T_x and T_z we use the boundary conditions on the surface of the plane. Thus, from relations (2.1), (2.2), (2.7), and (2.8) we obtain the system of algebraic equations

$$(u_+u_- + c_S^2\cos^2\alpha)\,T_x + c_S^2T_z\sin\alpha\cos\alpha = -\rho_0 u_0 u_+ u_- (u_+ + u_-)\sin\alpha \tag{2.9}$$

$$c_S^2T_x\sin\alpha\cos\alpha + (u_+u_- + c_A^2 + c_S^2\sin^2\alpha)\,T_z = \rho_0 u_0 u_+ u_- (u_+ + u_-)\cos\alpha \tag{2.10}$$

which has the simple solution

$$T_x = -\frac{\rho_0 u_0\sin\alpha}{u_+ + u_-}(u_+^2 + u_-^2 + u_+u_-) \tag{2.11}$$

$$T_z = \frac{\rho_0 u_0\cos\alpha}{u_+ + u_-}(u_+u_- + c_S^2) \tag{2.12}$$

The quantities T_x and T_z taken with the opposite signs yield the components of the radiation reaction force exerted by the medium on unit area of the oscillating surface. As Eqs. (2.11) and (2.12) reveal, the vector **T** does not coincide in direction with the normal **n** (Fig. 1). Exceptions are the special case $c_A = 0$, as well as $\alpha = 0$ and $\alpha = \pi/2$, in which case $\mathbf{T} = T\mathbf{n}$.

Expressions (2.11) and (2.12) are simplified in the two extreme cases of a hot medium, $c_S \gg c_A$, and a cold medium, $c_A \gg c_S$ [13]. Thus, for $c_S \gg c_A$ we have $u_+ \simeq c_S$, $u_- \approx c_A \cos\alpha$ (1.14), and

$$T_x \approx -\rho_0 c_S u_0 \sin\alpha\left(1 + \frac{c^2{}_A}{c^2{}_S}\cos\alpha\right), \quad T_z \approx \rho_0 c_S u_0 \cos\alpha \tag{2.13}$$

In the case $c_A \gg c_S$ we have $u_+ \simeq c_A$, $u_- \approx c_S \cos\alpha$, and

$$T_x \approx \rho_0 u_0 c_A \sin\alpha\left(1 - \frac{c_S{}^2}{c_A{}^2}\cos\alpha\right), \quad T_z \approx \rho_0 u_0 c_S \cos\alpha\left(\cos\alpha + \frac{c_S}{c_A}\right) \tag{2.14}$$

Note that if $c_S \to 0$, then $T_z \to 0$.

Substituting T_x and T_z from Eqs. (2.11) and (2.12) into (2.7) and (2.8), we obtain final expressions for the velocity components:

$$v_x = -\frac{u_0 \sin\alpha}{u_+{}^2 - u_-{}^2}\left[u_+{}^2 \exp(i\Psi_+) - u_-{}^2 \exp(i\psi_-)\right] \tag{2.15}$$

$$v_z = \frac{u_0 \cos\alpha}{u_+{}^2 - u_-{}^2}\left[(c_S{}^2 - u_-{}^2)\exp(i\psi_+) - (c_S{}^2 - u_+{}^2)\exp(i\psi_-)\right] \tag{2.16}$$

in which the following notation is introduced for the phase of the fast and slow magnetoacoustic waves:

$$\psi_\pm = -\omega_0\left(t + (x\sin\alpha - z\cos\alpha)/u_\pm\right) \tag{2.17}$$

From relations (1.2)-(1.4) and Eqs. (2.15)-(2.17) we obtain

$$\rho = \frac{\rho_0 u_0}{u_+{}^2 - u_-{}^2}\left[\frac{u_+{}^2\sin^2\alpha - (u_-{}^2 - c_S{}^2)\cos^2\alpha}{u_+}\exp(i\psi_+) - \frac{u_-{}^2\sin^2\alpha - (u_+{}^2 - c_S{}^2)}{u_-}\exp(i\psi_-)\right] \tag{2.18}$$

$$h_x = H_0 \frac{u_0 \sin 2\alpha}{2(u_+{}^2 - u_-{}^2)}\left[u_+ \exp(i\psi_+) - u_- \exp(i\psi_-)\right] \tag{2.19}$$

$$h_z = H_0 \frac{u_0 \sin^2\alpha}{u_+{}^2 - u_-{}^2}\left[u_+ \exp(i\psi_+) - u_- \exp(i\psi_-)\right] \tag{2.20}$$

$$v_y = h_y = 0, \quad p = c_S{}^2\rho \tag{2.21}$$

It is important to bear in mind that the linear equations (1.1)-(1.4) were obtained under the condition $|h|/H_0 \ll 1$, $(\rho/\rho_0) \ll 1$. On the basis of (2.18)-(2.20) these inequalities assume the form $|h/H_0| \le (u_0/u_+) \ll 1$ and, analogously, $|\rho/\rho_0| \le (u_0/u_+) \ll 1$.

Consequently, a necessary condition for linearization is $(u_0/u_+) \ll 1$, which is the same as the familiar condition in acoustics, $u_0 \ll c_S$ for $c_S \gg c_A$. We also point out that real pistons have finite dimensions for the oscillating surface. We have assumed here, by analogy with acoustical theory, that the maximum radiated wavelength $\lambda_+ = (2\pi u_+/\omega) \ll l$, where l is the minimum dimension of the piston surface.

The radiation intensity is determined according to Eqs. (1.8) and (2.11)-(2.12):

$$I = \frac{\rho_0 u_0{}^2}{2(u_+ + u_-)}\left[c_A{}^2\sin^2\alpha + c_S{}^2 + u_+ u_-\right] \tag{2.22}$$

In the derivation of (2.9)-(2.22) we used the simple relations

$$u_+{}^2 + u_-{}^2 = c_A{}^2 + c_S{}^2, \quad u_+ u_- = c_A c_S \cos\alpha$$

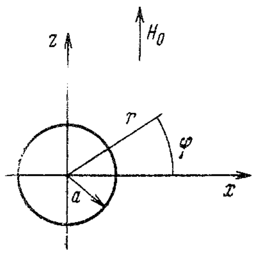

Fig. 2

which follow from (1.14). For the two extreme orientations of the oscillating plane we have according to (2.22)

$$I = \begin{cases} \rho_0 c_S u_0^2 / 2 & \text{for } \alpha = 0 \\ \rho_0 \sqrt{c_A^2 + c_S^2}\, u_0^2 / 2 & \text{for } \alpha = \pi/2 \end{cases} \tag{2.23}$$

Therefore, if the normal $\mathbf{n} \parallel \mathbf{H}_0$, ordinary sound is generated; but if $\mathbf{n} \perp \mathbf{H}_0$, a fast wave is radiated with a phase velocity $u_+ = (c_A^2 + c_S^2)^{1/2}$. Thus, the problem of the radiation of sound by an oscillating plane is solved for any values of the velocities c_A and c_S.

3. Radiation of Magnetic Sound by a Pulsating Cylinder

For more complex radiators, such as a radially pulsating cylinder or sphere, the radiation problem cannot be solved for arbitrary values of c_A and c_S. We therefore limit the present discussion to the generation of so-called "magnetic sound" in a cold medium, i.e., for $c_S = 0$ [13]. The other extreme case, $c_A = 0$, $c_S \neq 0$, has been thoroughly investigated in acoustics, and we shall not discuss it further.

Let us consider the generation of magnetic sound by a circular cylinder whose axis is directed along the y axis (Fig. 2), i.e., perpendicular to the force lines of H_0. The velocity perturbations due to radial pulsations of the cylinder are again determined by Eqs. (1.1)-(1.16) for $c_S = 0$:

$$\rho_0 v_x = i\omega \iint\limits_{-\infty}^{+\infty} \frac{f_{2x}}{\omega^2 - c_A^2 (k_x^2 + k_z^2)} e^{-ik_x x - ik_z z}\, dk_x\, dk_z; \quad v_y = 0 \tag{3.1}$$

$$\rho_0 v_z = \frac{i}{\omega} \iint\limits_{-\infty}^{+\infty} f_{2z} e^{-ik_x x - ik_z z}\, dk_x\, dk_z \tag{3.2}$$

Once again we omit the time factor $\exp(-i\omega t)$. The boundary condition (1.7) on the surface of the cylinder now has the form

$$v_x = u_0 \cos\varphi, \quad v_z = u_0 \sin\varphi \quad \text{for} \quad r = a \tag{3.3}$$

where u_0 is the velocity amplitude of the radial oscillations of the surface of the cylinder, $u_0 = a_1\omega$, and a_1 is the radial displacement, which we assume to be small in comparison with the average radius of the cylinder: $a_1 \ll a_0$. Throughout the ensuing analysis we adopt as the independent variables the polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ (Fig. 2). We represent the distribution of the force density **f** in the form

$$\mathbf{f} = \frac{T_x(\varphi)\mathbf{e}_x + T_z(\varphi)\mathbf{e}_z}{\sqrt{ra}} \delta(r - a) \tag{3.4}$$

where $T_x(\varphi)$ and $T_z(\varphi)$ are the components of the force exerted on the medium by unit length of the cylinder. These unknown functions of the angle φ are to be determined by means of the boundary conditions (3.3) and relations (3.1) and (3.2).

It is a particularly simple matter to determine $T_z(\varphi)$. Thus, we have by virtue of (3.2) and (3.3)

$$T_z(\varphi) = -i\rho_0 u_0 \omega \cos\varphi \lim_{r\to a} \sqrt{ra} / \delta(r - a) = 0 \tag{3.5}$$

because by definition $\lim \delta(r - a) = \infty$ as $r \to a$.* To determine the distribution of $T_x(\varphi)$ we substitute $\mathbf{f}_x$ (3.4) into the integral (1.18):

$$f_{2x} = \frac{1}{4\pi^2} \int_0^{2\pi} T_x(\varphi) e^{ika\cos(\varphi - \varphi')} d\varphi \quad (k_x = k\cos\varphi', \quad k_z = k\sin\varphi') \tag{3.6}$$

We use the familiar representation

$$e^{ika\cos(\varphi-\varphi')} = \sum_{n=-\infty}^{+\infty} i^n J_n(ka) e^{in(\varphi-\varphi')} \tag{3.7}$$

*See [14, 15] with regard to the radial Dirac delta function $\delta(r - a)$ and other delta-function representations.

in which J_n is a Bessel function of the first kind of order n. From relations (3.6) and (3.7) we find

$$f_{2x}(k, \varphi') = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} i^n T_{xn} J_n(ka) e^{-in\varphi'} \tag{3.8}$$

$$T_{xn} = \frac{1}{2\pi} \int_0^{2\pi} T_x(\varphi) e^{in\varphi} d\varphi, \quad T_x = \sum_{n=-\infty}^{+\infty} T_{xn} e^{-in\varphi} \tag{3.9}$$

We substitute relation (3.8) into the primary integral (3.1) and once again, making use of the representation (3.7), find

$$\rho_0 v_x(r, \varphi) = i\omega \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} T_{xn} e^{\frac{i(n-m)\pi}{2} - im\varphi} \left[\frac{1}{2\pi} \int_0^{2\pi} e^{-i(n-m)\varphi'} d\varphi'\right] \int_0^{\infty} \frac{k J_n(ka) J_n(kr)\, dk}{\omega^2 - c_A^2 k^2} \tag{3.10}$$

The integral on φ' gives the Kronecker delta function δ_{nm}, which permits summation over m to be carried out in Eq. (3.10). Then the second integral, which contains a product of Bessel functions, goes over to the well-known integral [16]

$$\int_0^{\infty} \frac{k J_n(ka) J_n(kr)}{\omega^2 - c_A^2 k^2} = -\frac{\pi i}{2c_A^2} J_n(k_0 a) H_n^{(1)}(k_0 r) \quad \text{for} \quad r \geqslant a \tag{3.11}$$

in which $H_n^{(1)}$ is a Hankel function of the first kind and $k_0 = \omega/c_A$ is the wave number. Therefore, expression (3.10) assumes the form

$$\rho_0 v_x(r, \varphi) = \frac{\pi\omega}{2c_A^2} \sum_{n=-\infty}^{+\infty} T_{xn} J_n(k_0 a) H_n^{(1)}(k_0 r) e^{-in\varphi} \tag{3.12}$$

If we substitute the expression (3.9) for T_{xn} into (3.12) and use the addition theorem for cylinder functions, we obtain the integral representation

$$\rho_0 v_x(r, \varphi) = \frac{\omega}{4c_A^2} \int_0^{2\pi} T_x(\varphi') H_0^{(1)}\left[k_0 \sqrt{r^2 + a^2 - 2ar\cos(\varphi - \varphi')}\right] d\varphi'$$

From (3.12), using the boundary condition (3.3), we find

$$T_{xn} = \frac{\delta_{n,1} + \delta_{n,-1}}{\pi k_0 J_n(k_0 a) H_n^{(1)}(k_0 a)} \rho_0 u_0 c_A \tag{3.13}$$

and on the basis of (3.9) we finally obtain

$$T_x = \frac{2\rho_0 u_0 c_A \cos\varphi}{\pi k_0 J_1(k_0 a) H_1^{(1)}(k_0 a)} \tag{3.14}$$

The perturbations in the medium are easily determined with the help of (1.2)–(1.4), (3.12), and (3.14):

$$v_x = u_0 \cos\varphi \frac{H_1^{(1)}(k_0 r)}{H_1^{(1)}(k_0 a)}, \quad \rho = i\rho_0 \frac{u_0}{c_A} \frac{k_0 r \cos^2\varphi H_1^{(1)\prime}(k_0 r) + \sin^2\varphi H_1^{(1)}(k_0 r)}{k_0 r H_1^{(1)}(k_0 a)}$$

$$h_x = iH_0 \frac{u_0}{c_A} \sin\varphi\cos\varphi \frac{k_0 r H_1^{(1)\prime}(k_0 r) - H_1^{(1)}(k_0 r)}{k_0 r H_1^{(1)}(k_0 a)}$$

$$h_z = H_0 \frac{\rho}{\rho_0}, \quad v_z = v_y = 0, \quad h_y = 0, \quad p = 0 \tag{3.15}$$

Thus, with regard for the factor $\exp(-i\omega t)$ it is apparent that relations (3.15) describe for $k_0 r \gg 1$ cylindrical magnetoacoustic waves diverging from the cylinder, because the asymptotic behavior of the function $H_1^{(1)}$ and its derivative has the form $r^{-1/2}\exp(ik_0 r)$ [16].

The distribution of the radiation intensity with respect to the polar angle is found according to Eqs. (1.9) and (3.14):

$$\frac{dI}{d\varphi}=\frac{\rho_0 u_0^2 c_A \cos^2\varphi}{\pi k_0\,[J_1^2(k_0a)+N_1^2(k_0a)]} \tag{3.16}$$

where N_1 is a Neumann function. Consequently, the directivity pattern in the xz plane has a dipole character. The radiation is zero along the dipole axis $\varphi = \pi/2$. The total intensity

$$I=\frac{\rho_0 u_0^2 c_A}{k_0\,[J_1^2(k_0a)+N_1^2(k_0a)]} \tag{3.17}$$

has the simple asymptotic behavior

$$I=\begin{cases} {}^1\!/_4\pi^2\rho_0 u_0^2 a^2\omega & \text{for} \quad k_0a\ll 1 \\ {}^1\!/_2\pi\rho_0 u_0^2 c_A a & \text{for} \quad k_0a\gg 1 \end{cases} \tag{3.18}$$

Relations (3.18) are analogous to the corresponding equations for the intensity of ordinary sound radiated by a pulsating cylinder [9, 17]. The principal distinction of magnetic from ordinary sound is the fact that its waves have zero pressure perturbations, $p = 0$, and the velocity perturbations are anisotropic. In place of hydrodynamic pressure, magnetic sound has the magnetic pressue due to the presence of the magnetic field perturbations. The dipolar directivity pattern (3.16) is a consequence of the magnetic anisotropy of the properties of the medium.

4. Radiation by a Pulsating Sphere

The solution of the problem of the radiation of magnetic sound by a pulsating sphere is analogous to the preceding, and we can therefore omit the intermediate calculations. To simplify the analysis we assume that the radial oscillations of the sphere are axisymmetric about the z axis, which is aligned with the force lines of the magnetic field $\mathbf{H}_0$. Now Alfvén waves are not excited, and the condition $\mathrm{rot}_z f = 0$ (rot = curl) or, in k representation, $k_x f_{1y} = k_y f_{1x}$ holds, which in the event of an axisymmetric distribution of the velocity field on sphere is strictly deduced from the general relations (1.11)-(1.13) for any c_S and c_A (see also [5]). From the general relations (1.11)-(1.13) under this condition and the assumption $c_S = 0$ we obtain

$$\rho_0 v_{x,y}=i\omega\iiint\limits_{-\infty}^{+\infty}\frac{f_{1x,y}e^{-i(\mathbf{kr})}\,d\mathbf{k}}{\omega^2-c_A^2(k_x^2+k_y^2+k_z^2)} \tag{4.1}$$

$$\rho_0 v_z = if_z/\omega \tag{4.2}$$

The distribution of the forces f on the surface can be represented in the form

$$\mathbf{f}=[T_x(\varphi,\theta)\mathbf{e}_x+T_y(\varphi,\theta)\mathbf{e}_y+T_z(\varphi,\theta)\mathbf{e}_z]\,\delta(R-R_0)/RR_0 \tag{4.3}$$

The independent variables in this expression are the spherical coordinates R, θ, φ with polar axis along z; R_0 is the radius of the sphere. On its surface the boundary condition (1.8) for homogeneous radial pulsations is written in the form

$$v_x=u_0\sin\theta\cos\varphi,\quad v_y=u_0\sin\theta\sin\varphi,\quad v_z=u_0\cos\theta \tag{4.4}$$

where $u_0 = \omega R_1$ is the velocity amplitude of the radial oscillations of the sphere, and we assume here that $R_1 \ll R_0$. The time factor $\exp(i\omega t)$ is omitted everywhere. As in the case of the pulsating cylinder, we find the following at once from Eqs. (4.2)-(4.4):

$$T_z(\varphi,\theta)=\lim_{R\to R_0}\frac{\omega\rho_0 u_0 R R_0\cos\theta}{i\delta(R-R_0)}=0 \tag{4.5}$$

For the analysis of the integral (4.1) we introduce the spherical coordinates

$$k_z=q\cos\theta'\quad k_x=q\sin\theta'\cos\varphi',\quad k_y=q\sin\theta'\sin\varphi'$$

and use the following expansion of the exponential function:

$$\exp(-iqR\cos\psi) = \left(\frac{\pi}{2qR}\right)^{1/2} \sum_{n=0}^{\infty} i^n (2n+1) J_{n+1/2}(qR) P_n(\cos\psi) \tag{4.6}$$

$$\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$$

Here $J_{n+1/2}$ are Bessel functions, and P_n are Legendre polynomials, for which the addition theorem yields

$$P_n(\cos\psi) = \sum_{m=-n}^{+n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') e^{im(\varphi-\varphi')} \tag{4.7}$$

It is clear from the foregoing that the unknown functions $T_{x,y}(\varphi, \theta)$ are conveniently sought in the form of an expansion in sphere functions P_n^m:

$$T_{x,y} = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} T_{x,y}(n, m) P_n^m(\cos\theta) e^{im\varphi} \tag{4.8}$$

From Eqs. (1.19), (4.6)-(4.8) we find

$$f_{1,x,y} = \frac{1}{2\pi^2} \left(\frac{\pi}{2qR_0}\right)^{1/2} \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} i^n T_{x,y}(n, m) J_{n+1/2}(qR_0) P_n^m(\cos\theta') e^{im\varphi'} \tag{4.9}$$

Substituting (4.9) into the integral (4.1), we obtain

$$\rho_0 v_{x,y}(R, \varphi, \theta) = \frac{\pi\omega}{2C_A^2 \sqrt{RR_0}} \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} T_{x,y}(n, m) J_{n+1/2}(k_0R_0) H^{(1)}_{n+1/2}(k_0R_0) P_n^m(\cos\theta) e^{im\varphi} \tag{4.10}$$

where $k_0 = \omega/c_A$. Equation (4.10) is exactly analogous to (3.12). We can use (4.10) to determine $T_{x,y}$ for rather arbitrary boundary conditions on the surface of the sphere, as long as they are, as mentioned, axisymmetric about the z axis. In the case of homogeneous radial pulsations of the sphere (4.4) we find the following from relations (4.8) and (4.10):

$$T_{x,y} = \frac{2\rho_0 u_0 c_A^2 R_0 \sin\theta}{\pi\omega J_{3/2}(k_0R_0) H^{(1)}_{3/2}(k_0R_0)} \begin{Bmatrix}\cos\varphi \\ \sin\varphi\end{Bmatrix} \tag{4.11}$$

and the velocity perturbations have the form

$$v_{x,y} = u_0 \sin\theta \frac{H^{(1)}_{3/2}(k_0R)}{H^{(1)}_{3/2}(k_0R_0)} \begin{Bmatrix}\cos\varphi \\ \sin\varphi\end{Bmatrix}, \quad v_z = 0 \tag{4.12}$$

In Eqs. (4.11) and (4.12) $\cos\varphi$ refers to the x components. The magnetoacoustic radiation intensity in an element of solid angle $d\Omega = \sin\theta\, d\theta d\varphi$ is determined by means of (4.11) according to Eq. (1.8):

$$\frac{dI}{d\Omega} = \frac{\rho_0 u_0^2 c_A^2 R_0}{\pi\omega [J^2_{3/2}(k_0R_0) + N^2_{3/2}(k_0R_0)]} \sin^2\theta \tag{4.13}$$

Consequently, the directivity pattern for the radiation of magnetic sound by a radially pulsating sphere is dipolar, by contrast with the acoustic case $c_A = 0$, $c_S \neq 0$, when the same sphere generates isotropic radiation (acoustic monopole) [9]. The total radiation intensity is

$$I = \frac{4\pi\rho_0 u_0^2 c_A k_0^2 R_0^4}{3(1 + k_0^2 R_0^2)} \tag{4.14}$$

Here we have relied on the well-known formulas

$$J_{3/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\left(\frac{\sin x}{x} - \cos x\right), \quad N_{3/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\left(\sin x + \frac{\cos x}{x}\right) \tag{4.15}$$

Thus, in cold magnetoacoustic media, in which the magnetic pressure $H_0^2/4\pi$ greatly exceeds the hydrodynamic pressure p_0, the latter condition being equivalent to $c_A \gg c_S$, the mechanical motion of solids generates magnetic sound. The latter radiation differs significantly from ordinary sound generated by the same radiators in cold media, $c_S \gg c_A$. This conclusion is also true for more complex radiators than those considered here. We also point out that the results obtained above [see (3.12) snd (4.10)] can be used as a basis for analyzing the generation of magnetic sound by cylinders and spheres having more complex pulsation velocity distributions over their surfaces, by analogy with the familiar problems of acoustics [8].

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